Math 206B Lecture 14 Notes

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February 8, 2019

1 Applications of The Hilman-Grassl Bijection

1.1 MacMahon's theorem

We showed the following last time:

Theorem 1.1 (Stanley)**.**

$$\sum_{A \in \mathrm{RPP}(\lambda)} t^{|A|} = \prod_{(i,j) \in \lambda} \frac{1}{1 - t^{h_{i,j}}},$$

where $h_{i,j}$ is the hook length of the square i, j in the Young diagram of λ.

Definition 1.1. A **plane partition** of shape λ is a tableau which is weakly decreasing going down or to the right.

Example 1.1. Here is a plane partition of shape $(4, 4, 3, 2)$.

7	7	6	4
4	3	1	1
3	3	1	
2			

Theorem 1.2 (MacMahon, c. 1900)**.**

$$\sum_{A \in \mathrm{PP}} t^{|A|} = \prod_{k=1}^{\infty} \frac{1}{(1 - t^k)^k}$$

Proof. Let $\lambda = m^m$; this is a square diagram. Then Stanley's theorem says

$$\sum_{A \in \mathrm{RPP}(m^m)} t^{|A|} = \prod_{k=1}^{m} \frac{1}{(1 - t^k)^k} \cdot \prod_{k=m+1}^{2m-1} \frac{1}{(1 - t^k)^{2m-k}}.$$

Now

$$\sum_{A \in PP} t^{|A|} = \lim_{m \to \infty} \sum_{A \in PP \cap (m^m)} t^{|A|}$$

because this limit stabilizes for each coefficient of the power series.

$$\begin{aligned}
&= \lim_{m \to \infty} \sum_{A \in RPP(m^m)} t^{|A|} \\
&= \lim_{m \to \infty} \prod_{k=1}^{m} \frac{1}{(1-t^k)^k} \cdot \prod_{k=m+1}^{2m-1} \frac{1}{(1-t^k)^{2m-k}} \\
&= \prod_{k=1}^{m} \frac{1}{(1-t^k)^k}.
\end{aligned}$$

□

MacMahon's theorem is analogous to the following result.

Theorem 1.3 (Euler, 1738)**.**

$$\sum_{\lambda \in P} t^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1-t^k}.$$

1.2 The hook length formula

We can also prove the hook length formula using Stanley's theorem.

Theorem 1.4 (Frame-Robinson-Thrall, 1954)**.**

$$f^\lambda = \#\,\mathrm{SYT}(\lambda) = n! \prod_{(i,j) \in \lambda} \frac{1}{h_{i,j}}.$$

Example 1.2. Suppose $\lambda = (m, m)$. Here is a standard Young tableau of this shape:

1	2	4	5	9
3	6	7	8	10

Then the hook lengths of each square look like

$m+1$	m	$\cdots$	3	2
m	$m-1$	$\cdots$	2	1

In general,

$$f^{(m,m)} = \frac{(2m)!}{m!(m-1)!} = \frac{1}{m+1}\binom{2m}{m} = \mathrm{Cat}(m),$$

the m-th Catalan number.

Proof. Write

$$\mathrm{RPP}(\lambda) = \bigcup_{A \in \mathrm{SYT}(\lambda)} C_A,$$

where C_A is where we pick the numbers in order of sums of indices in the diagram. That is $0 \le x_{1,1} \le x_{1,2} \le x_{2,1} \le x_{1,3} \le x_{2,2} \le x_{1,4} \le x_{2,3} \le x_{3,1} \le \cdots$; this is a cone in $\mathbb{R}^n$. Now look at the number of $A \in \mathrm{RPP}(\lambda)$ such that $|A| \le N$. Asymptotically, this is about

$$\begin{aligned}
\sum_{T \in \mathrm{SYT}(\lambda)} \#\{A \in C_T : |A| \le N\} &= \#\,\mathrm{SYT}(\lambda) \cdot \#\{0 \le z_1 \le \cdots \le z_n : z_1 + \cdots + z_n \le N\} \\
&\sim \#SYT(\lambda) \cdot N^n \operatorname{vol}(\Delta),
\end{aligned}$$

where $\Delta = \{0 \le z_1 \le z_2 \le \cdots \le z_n : z_1 + \cdots + z_n \le 1\}$. The vertices of Δ are when we have equalities. So we have

$$\begin{aligned}
v_0 &= (0, 0, \ldots, 0) \\
v_1 &= (0, \ldots, 0, 1) \\
v_2 &= (0, \ldots, 0, 1/2, 1/2) \\
v_3 &= (0, \ldots, 0, 1/3, 1/3, 1/3) \\
&\vdots \\
v_n &= (1/n, \ldots, 1/n).
\end{aligned}$$

This is a triangular matrix, so the volume ($1/n!$ times the determinant), is $1/n!$ times the product of the diagonal entries. That is,

$$\mathrm{Vol}(\Delta) = \frac{1}{n!} \cdot \frac{1}{n!}.$$

So we get

$$|\{A \in \mathrm{RPP}(\lambda) : |A| \le N\}| \sim \frac{N^n}{(n!)^2} \#\,\mathrm{SYT}(\lambda).$$

The right hand side of Stanley's theorem is

$$\sum t^{\sum b_{i,j} h_{i,j}},$$

where the sum is over matrices $B = (b_{i,j})$ such that $b_{i,j} \ge 0, \sum b_{i,j} \le N$. This is asymptotically $N^n \operatorname{vol}(\Delta')$, where $\Delta' = \{0 \le y_{i,j}, \sum y_{i,j} h_{i,j} \le 1\}$. The vertices of Δ' are

$$\begin{gathered}
(0, 0, \ldots, 0) \\
(1/h_{1,1}, 0, \ldots, 0) \\
(0, 1/h_{1,2}, 0, \ldots, 0)
\end{gathered}$$

$$\vdots$$

So we get that

$$\mathrm{Vol}(\Delta') = \frac{1}{n!} \cdot \prod_{(i,j)\in\Delta} \frac{1}{h_{i,j}}.$$

By Stanley's theorem, we have

$$|\{A \in \mathrm{RPP}(\lambda) : |A| \le N\| = |\{B : b_{i,j} \ge 0, \sum b_{i,j} \le N\}|$$

So the asymptotics have to be the same. Then we get that

$$\#\,\mathrm{SYT}(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}. \qquad \square$$

We will prove the hook length formula in different ways, as well.